Probability and Statistics, Sample Prelim II Questions, Fall 2021

1. Suppose the random variable X has pdf

$$f(x;\theta) = \begin{cases} \theta X^{-\theta-1} & \text{for } x \geq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the Jeffrey's prior $\Pi(\theta)$

(b) Find the MLE of θ and the Fisher's information.

(c) Find the 95% CI for θ based on a sample of size $n = 100$, with $\prod_{i=1}^{n} X_i = 40320$.

(d) Find the 95% CI for $\sqrt{\theta}$ based on Delta method.

2. Let X_k have Gamma distribution with $\alpha = k$, k is an integer, and $\beta = \theta$.

(a) Show that X_k/k converges to θ as $k \to \infty$.

(b) Show that

$$\frac{X_k - k\theta}{\theta\sqrt{k}}$$

converges in distribution to a standard normal random variable as $k \to \infty$

(c) Show that

$$\frac{\sqrt{k}\,(X_k/k - \theta)}{X_k/k}$$

converges in distribution to a standard normal random variable as $k \to \infty$

(d) Assuming that k and X_k are known, use the result in (c) to derive an approximate (large k) confidence interval for θ.

3. Suppose that $X_1, X_2, \ldots, X_n$ is a sample from a distribution with density function

$$f_\theta(y) = \begin{cases} (\theta+1)\theta^y, & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > -1$.

(a) Find an estimator $\tilde{\theta}$ for θ by the method of moments.

(b) Find an estimator $\hat{\theta}$ for θ by the method of maximum likelihood.

(c) Compute the bias and variance of each estimator. Which estimator would you prefer and why?

4. Suppose that $X_1, X_2, \ldots, X_n$ is a sample from the exponential distribution with the density function $f(x;\theta) = \theta e^{-\theta x}, \quad x > 0$. Assume a prior density on θ which is also exponential with the mean $1/\beta$, β is known.

(a) Prove that the posterior distribution is a Gamma distribution and find its parameters.
IF you cannot do part (a), assume the posterior distribution is a Gamma distribution with parameters a, b and do the remaining parts.

(b) Using the squared error loss, find the Bayes estimator of θ.

(c) Using the absolute error loss, find the Bayes estimator of θ (this won't have an explicit analytical expression but your answer can be expressed through a percentile of the Gamma distribution).

(d) Derive a 95% Bayesian credible interval for θ.

(e) Derive a 95% Bayesian credible interval for $\mu = \frac{1}{\theta}$.

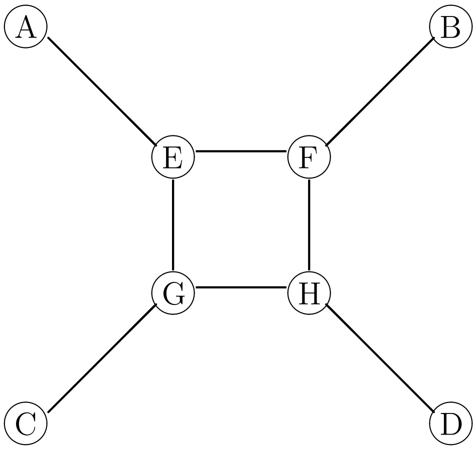
5. A Markov chain is defined by a random walk on the graph pictured below. From a given node, you are equally likely to go to any neighboring node.

(a) Specify the transition matrix.

(b) Find the stationary distribution for this Markov chain.

(c) Find the probability, when starting from A, to visit C before you visit D.

(d) Find the expected time, when starting from A, to visit C.



6. Consider a Poisson process $X(t)$ with intensity λ, so that

$$p_k(t) = P(X(t) = k) = \exp(-\lambda t)\frac{(\lambda t)^k}{k!}.$$

Another Poisson process $Y(t)$, independent of the first one, has the intensity ν. Show that $X(t) + Y(t)$ is also a Poisson process, and find its intensity.

7. Let $X(t)$ be a pure death process with initial value $X(0) = N$ and the death rate $\mu_n = n\mu, n = N, N-1, ..., 1$. Let $P_n(t) = P(X(t) = n)$. Find a system of differential equations for $P_n(t)$ and show that their solution is

$$P_n(t) = \binom{N}{n} e^{-n\mu t}(1 - e^{-\mu t})^{N-n}.$$