Probability and Statistics, Sample Prelim III Questions, Fall 2021

1. For the Normal random variable with density

$$f(x;\mu\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

consider estimating the parameter $\boldsymbol{\theta} = (\mu, \sigma)'$.

(a) Find the Fisher information matrix for $\boldsymbol{\theta}$

(b) Using the multivariate Delta-method, find the approximate distribution for $\tau = \mu + \theta$.

(c) Find a large-sample 99% confidence interval for τ.

2. Let X be uniformly distributed between 1 and 3. Let Y, conditioned on X, be exponentially distributed with the rate $\lambda = X$. Find $\mathbb{E}(Y)$. Find $P(X > 5)$.

3. Suppose that $\mathbb{E}(\hat{\theta}_1) = \mathbb{E}(\hat{\theta}_2) = \theta$, $Var(\hat{\theta}_1) = \sigma_1^2$ and $Var(\hat{\theta}_2) = \sigma_2^2$. Consider the estimator $\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2$.

(a) Show that $\hat{\theta}_3$ is an unbiased estimator of θ.

(b) If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, how should constant a be chosen in order to minimize the variance of $\hat{\theta}_3$?

(c) If $Cov(\hat{\theta}_1, \hat{\theta}_2) = c \neq 0$, how should constant a be chosen in order to minimize the variance of $\hat{\theta}_3$?

4. Consider the following joint density for random variables X and Y:

$$f(x,y) = \begin{cases} kxy, & \text{for } 0 < x < 1, x < y < 2-x \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find k.

(b) Find the marginal density of Y. Are these random variables independent?

(c) Derive the conditional density $f(x|Y=y)$, also find the conditional mean and conditional variance of X given $Y = y$.

(d) Find $Cov(X,Y)$.

5. A single observation X is taken from a population with density function

$$f(x, \theta) = \theta e^{-\theta x}, \quad \text{for } x > 0 \text{ and } \theta > 0.$$

(a) For $c > 0$ consider the test that rejects $H_0 : \theta = \theta_0$ in favor of $H_1 : \theta = \theta_0$ when $X > c$. Determine c so that this is a size α test where $0 < \alpha < 1$.

(b) Invert the test in (a) to obtain a $1 - \alpha$ size confidence set for θ.

6. For a Markov Chain representing a random walk on the non-negative integers,

$$\begin{cases} X_{n+1} = X_n + 1, \text{with probability } p \\ X_{n+1} = X_n - 1, \text{with probability } q = 1 - p \end{cases}$$

unless $X_n = 0$, in which case

$$\begin{cases} X_{n+1} = 1, \text{with probability } p \\ X_{n+1} = 0, \text{with probability } q = 1 - p \end{cases}$$

(a) Give the conditions (with proof) under which the stationary distribution exists, then calculate the stationary distribution.

(b) Calculate the expected time, starting from 0, to reach the state 3.

7. For the Poisson process with the rate λ. Let W_k be the time when kth event occurs.

(a) Find the correlation coefficient between W_3 and W_4.

(b) Find the correlation coefficient between W_3 and W_4, under the condition that $W_5 = 10$

8. The customers are coming to the system according to the Poisson process with the rate λ. Each customer is immediately being served, with the service time being Exponential with the rate parameter μ. Let $X(t)$ be the total number of customers in the system at the tie t (currently being served).

(a) Model $X(t)$ as a birth-and-death process (specify the birth and death rates).

(b) Find the stationary distribution for $X(t)$, specify the conditions under which it exists.