Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
January 11, 2019

1. This exam is four hours long.
2. You need a scientific calculator for this exam.
3. Work out all six problems.
4. Start the solution of each problem on a new page.
5. Number all of your pages.
6. Sign your name on the following line and put the total number of pages.
7. Use this sheet as a coversheet for your papers.

NAME: ______________________________ **No. of pages:**____________

Problem 1.

Let $p(x)$ be the Lagrange interpolant of function $f(x)$ on the partition $\{x_i\}_{i=0}^n$. Let

$$\omega_i = \prod_{j=0, j\neq i}^{n} \frac{1}{(x_i - x_j)}, \quad i = 0, 1, \dots, n,$$

be the barycentric coefficients of the partition. Show that the interpolant can be evaluated using the formula

$$p(x) = \frac{\sum_{i=0}^n f(x_i) \frac{\omega_i}{x - x_i}}{\sum_{i=0}^n \frac{\omega_i}{x - x_i}}.$$

Hint: use a function $\psi(x) = \prod_{j=0}^n (x - x_j)$, and show that $\psi(x) = \left(\sum_{i=0}^n \frac{\omega_i}{x - x_i} \right)^{-1}$.

Problem 2.

You're given a positive real number a. Using Newton's method, develop an iterative procedure for computing $\sqrt[3]{a}$. Your method should use only the basic operations of addition, subtraction, multiplication, and division. Also find a starting point x_0 from which the method is guaranteed to converge.

Hint: use a theorem on convergence of a monotonic sequence.

Problem 3.

A clamped cubic spline $s(x)$ for a function $f(x)$ defined on the interval $[1, 3]$ is given by

$$s(x) = \begin{cases} 3(x-1) + a(x-1)^2 - (x-1)^3, & 1 \le x < 2, \\ 4 + b(x-2) + c(x-2)^2 + \frac{1}{3}(x-2)^3, & 2 \le x \le 3. \end{cases}$$

Find $f'(3)$.

Problem 4.

Let $Q = (q_{ij})$ be an n by n real matrix with nonnegative entries such that

$$\sum_{j=1}^n q_{ij} < 1, \quad i = 1, 2, \dots, n.$$

1. Show that

$$\lim_{m\to\infty} Q^m = 0.$$

2. Assuming that the sequence

$$\{I + Q + \cdots + Q^m\}_{m=0}^\infty$$

is convergent, show that

$$I + Q + Q^2 + \ldots = (I - Q)^{-1}.$$

Problem 5.

Let $x, y \in R^n$ be such that $x \neq y$, and $\|x\|_2 = \|y\|_2$.

1. Find a Householder reflector Q such that $Qx = y$.
2. Prove that Q is symmetric and orthogonal.

Problem 6.

Consider a descent iterative method

$$x_{k+1} = x_k + \alpha_k p_k$$

for solving a linear system $Ax = b$ with a positive definite matrix A. Let

$$J(x) = x^t Ax - 2b^t x.$$

1. Assuming that α_k is obtained by exact line search and $p_k^t r_k \neq 0$, prove that

$$J(x_{k+1}) < J(x_k).$$

2. Vectors x and y are conjugate if $x^T Ay = 0$. Let $p_0, \ldots, p_k$ be nonzero, mutually conjugate vectors, and let

$$p_{k+1} = r_{k+1} - \sum_{i=0}^{k} c_{ik} p_i.$$

Prove that vectors $p_0, \ldots, p_{k+1}$ are linearly independent, and mutually conjugate if and only if $p_{k+1} \neq 0$, and

$$c_{jk} = \frac{p_j^t A r_{k+1}}{p_j^t A p_j}, \ \forall j \leq k.$$