Date: August 8, 2023 **Name:**

PhD Preliminary Examination in Analysis
Department of Mathematics
New Mexico Tech
2023

1. Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and $f : [a,b] \to \mathbb{R}$ be a Riemann-integrable function with respect to α on the interval $[a,b]$. Let

$$M = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad m = \inf_{x\in[a,b]} f(x).$$

Prove there exists a number $c \in [m,M]$ such that

$$\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x).$$

2. Let $(x_n)_{n=1}^{\infty}$ be a sequence defined by

$$\begin{aligned} &0 < x_1 < 1, \\ &x_{n+1} = 1 - \sqrt{1 - x_n}, \qquad n \geq 1. \end{aligned}$$

Prove that the sequence (x_n) is convergent and find its limit.

3. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that there is a real number α such that

$$|f'(x)| \leq \alpha |f(x)|$$

for all $x \in [0,1]$. Prove that if $f(0) = 0$ then $f(x) = 0$ for all $x \in [0,1]$.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined for $x \in [0,1]$ by

$$f_n(x) = \frac{1}{1+n^2x^2}.$$

Does the sequence $\{f_n\}$ uniformly converge on the interval $[0,1]$? Justify your answer.

5. Let C_R be the circle of radius R centered at the origin oriented counterclockwise in the complex plane. Let f and g be two functions analytic outside the circle C_R. Suppose that they have the following limits at infinity

$$\lim_{z\to\infty} f(z) = 0, \qquad \lim_{z\to\infty} zg(z) = 1.$$

Show that

$$\frac{1}{2\pi i}\oint_{C_R} g(z)e^{f(z)}\, dz = 1\,.$$

6. Let C be a closed contour without self-intersections oriented counterclockwise and w be a complex number. Let f be a complex function such that: (i) both f and f' are analytic inside and on C and (ii) $f(z) \neq w$ on the boundary of C. Let N be the number of points z inside C such that $f(z) = w$. Show that

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z) - w} dz = N$$

7. Let f be an entire function such that $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. Show that there are constants a and b such that $f(z) = a + bz^2$ with $|b| \leq \frac{1}{2}$.

8. Show that for $0 < p < 1$

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1 + e^x} dx = \frac{\pi}{\sin(p\pi)}$$