Date: April, 2024 **Name:**

PhD Preliminary Examination in Analysis

Department of Mathematics

New Mexico Tech

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1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$x_1 > \sqrt{2} \quad \text{and} \quad x_{n+1} = \frac{2 + x_n}{1 + x_n}, \quad n = 1, 2, \ldots$$

   Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges and find its limit.

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. Prove that f is integrable with respect to α on $[a, b]$.

3. Let $f : (0, 1) \to \mathbb{R}$ be a nonnegative smooth function such that it vanishes, $f(a) = f(b) = 0$, at two distinct interior points $a, b \in (0, 1)$. Prove that there exists a point $c \in (0, 1)$ such that $f'''(c) = 0$.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in \mathbb{R}$. Suppose that

$$f_1(t) \le f_2(t) \le f_3(t) \le \cdots \le f_n(t) \le \cdots, \quad \forall\, t \in \mathbb{R}.$$

   Prove for every sequence $\{t_k\}_{k=1}^{\infty}$ of real numbers converging to a real number t, that is, if $\lim_{k\to\infty} t_k = t$, there holds

$$\liminf_{k\to\infty} f(t_k) \ge f(t).$$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that if there is a real constant K such that

$$|f(z)| \le K|z| \qquad \text{for all } z \in \mathbb{C},$$

   then there is a real constant C such that $f(z) = Cz$ for all $z \in \mathbb{C}$.

6. Let $f : D \to \mathbb{C}$ be an analytic function in a region D. Show that if $f'(z_0) \neq 0$ at some interior point $z_0 \in D$, then f is not constant in a neighborhood of any point $z \in D$.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that if $\operatorname{Im} f(z) \le 0$ for any $z \in \mathbb{C}$ then f is constant.

8. Let $f : D \to \mathbb{C}$ be an analytic function in a compact region D. Show that the modulus $|f|$ has a minimum value in D that occurs on the boundary ∂D of the region D.

9. Let f be an entire function. Show that if

$$|f'(z)| \leq |z| \qquad \text{for all } z \in \mathbb{C},$$

then there are constants $a, b \in \mathbb{C}$ such that

$$f(z) = a + bz^2,$$

with $|b| \leq \frac{1}{2}$.

10. Let $P(z)$ be a polynomial of degree higher than 2 and f be a meromorphic function defined by

$$f(z) = \frac{1}{P(z)}.$$

Show that the sum of the residues of f at all the zeros of P is equal to 0.

11. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two entire functions. Suppose that:

    (a) the functions f and g have no zeros, and

    (b)
    $$\lim_{z\to\infty} \frac{f(z)}{g(z)} = 1\,.$$

Show that $f(z) = g(z)$ for any $z \in \mathbb{C}$.

12. Let P be a polynomial, C be a sufficiently large circle centered at the origin and oriented counterclockwise and

$$I = \frac{1}{2\pi i} \oint_C dz\, \frac{P'(z)}{P(z)}.$$

Show that I is equal to the degree of the polynomial P.

13. Let P be a polynomial, C be a sufficiently large circle centered at the origin and oriented counterclockwise and

$$I = \frac{1}{2\pi i} \oint_C dz\, z\frac{P'(z)}{P(z)}$$

Show that I is equal to the sum of the roots of the polynomial P counted with multiplicities.

14. Let $f : D \to \mathbb{C}$ be an analytic function in a simply connected domain D. Let C be a simple closed contour in D oriented counterclockwise and $a \in \mathbb{C}$ be a complex number such that $f(z) \neq a$ for any $z \in C$ and

$$I = \frac{1}{2\pi i} \oint_C dz \frac{f'(z)}{f(z) - a}.$$

Show that I is equal to the number of points inside C such that $f(z) = a$.

15. Let $f, g : D \to \mathbb{C}$ be two analytic functions on a region D. Suppose that f is injective (one-to-one) and $f'(z) \neq 0$ for any $z \in D$. Let C be a simple (without self-intersections) closed contour in D oriented counterclockwise and z_0 be a point inside C. Show that

$$\frac{1}{2\pi i} \oint_C dz \, \frac{g(z)}{f(z) - f(z_0)} = \frac{g(z_0)}{f'(z_0)} \, .$$

16. Let $f : D \to \mathbb{C}$ be an analytic function in a region D. Let z_0 be a point in D and C be a sufficiently small circle in D centered at z_0 oriented counterclockwise. Show that if $f'(z_0) \neq 0$, then

$$\frac{1}{2\pi i} \oint_C dz \, \frac{1}{f(z) - f(z_0)} = \frac{1}{f'(z_0)} \, .$$

17. Let C be the circle centered at the origin oriented counterclockwise. Let f, g be two functions analytic outside the circle C. Suppose that they have the following limits at infinity

$$\lim_{z\to\infty} f(z) = 0, \qquad \lim_{z\to\infty} z g(z) = 1.$$

Show that

$$\frac{1}{2\pi i} \oint_C g(z) e^{f(z)} \, dz = 1 \, .$$

18. Let C be the a circle of radius 8 centered at the origin oriented counterclockwise. Evaluate the integral

$$I = \frac{1}{2\pi i} \oint\limits_C dz \, \tan z \, .$$

19. Let C be the a sufficiently small simple closed contour not passing through the origin oriented counterclockwise. Evaluate the integral

$$I_n = \frac{1}{2\pi i} \oint\limits_C \frac{dz}{z} \left(z + \frac{1}{z} \right)^n ,$$

where n is an integer. Consider the cases $n > 0, n = 0, n < 0$.

20. Show that for $0 < p < 1$

$$\int_{-\infty}^{\infty} dx \, \frac{e^{px}}{1 + e^x} = \frac{\pi}{\sin(p\pi)}$$

21. Let sign $: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\operatorname{sign}(\omega) = \begin{cases} 1, & \text{if } \omega > 0 \, , \\ 0, & \text{if } \omega = 0 \, , \\ -1, & \text{if } \omega < 0 \, . \end{cases}$$

Show that

$$\int_0^{\infty} dx \, \frac{\sin(\omega x)}{x} = \frac{\pi}{2} \operatorname{sign}(\omega) \, .$$

22. Let p, q be two positive real numbers such that $0 < p < q$. Show that

$$\int_0^\infty dx \frac{x^{p-1}}{1+x^q} = \frac{\pi}{q} \frac{1}{\sin\left(\frac{p}{q}\pi\right)}$$

23. Let $\alpha, \beta \in \mathbb{R}$ be two real numbers. Use principal value integrals to show that

$$\int_0^\infty \frac{dx}{x^2} \left[\cos(\alpha x) - \cos(\beta x)\right] = -\frac{\pi}{2} \left(|\alpha| - |\beta|\right) .$$

24. Let $p, c \in \mathbb{R}$ be two positive real numbers such that $0 < p < 1$ and $c > 0$. Show that

$$\int_0^\infty dx \, \frac{x^{p-1}}{x+c} = c^{p-1} \frac{\pi}{\sin(p\pi)} .$$